On the Theory of Strongly Coupled Cavity Chains*

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Summary—A chain of identical cavity resonators coupled together through slots in their common walls forms a band-pass microwave filter. The pass band characteristics of such a system are determined by a combination of field theory and circuit theory. The fields in the cavities are expressed in terms of the normal modes of the uncoupled cavities. The fields in the neighborhood of a slot are determined by representing the slot as a transmission line. Irrotational components of the field in the cavities account for direct slot-to-slot coupling. The method successfully predicts both the dispersion characteristics and field distributions over large frequency ranges for many practical systems, such as slow-wave circuits for high-power traveling-wave tubes.

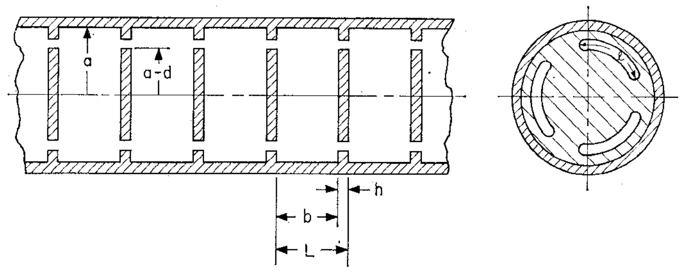

Fig. 1—Cylindrical cavity resonators coupled by long slots.

I. Introduction

The theoretical determination of the dispersion relations and the field distributions of a system of coupled cavity resonators is, in general, a complicated problem. Analytic relationships for the resonant frequencies of a pair of coupled cavities have been obtained by Bethe[1] and have been applied to the case of weak coupling between cavities through small circular apertures. His analysis was facilitated by the assumption that the aperture was small compared to the wavelength, so that he was able to assume that the fields in the region of the hole have a static distribution. In this paper, strong coupling will be treated for a system in which one dimension of the coupling aperture is comparable in length to a wavelength. Chains of strongly-coupled cavity resonators are used in practice as slow-wave circuits for high-power traveling-wave tubes. In particular, the case of a chain of cavities coupled by long narrow slots in their common walls will be considered. A typical system with which we shall deal is shown in Fig. 1. For interaction with an electron beam, a beam-passage hole would be cut in the common walls. The amount of coupling introduced by such a hole is small compared with the coupling through the slots.

The case of coupling by long narrow slots has been treated in the past by assuming a quasi-static field distribution across the narrow dimension of a slot. In order to find the coupling through a slot cut in the common wall of a pair of waveguides, Stevenson[2] used this assumption and the further one of a sinusoidal voltage distribution along the slot; because of this latter assumption his analysis applies only near the slot resonance. Akhiezer and Lyubarsky,[3] who considered cavities coupled by long narrow slots, also employed the electrostatic assumption for a narrow slot, but derived the field distribution along the length of the slot by employing the theory of slot antennae. Each of the two preceding formulations is long, quite involved, and difficult to use in practice. In this paper a new method of approach will be given which makes use of transmission-line theory to determine the excitation of a slot in a connecting wall which can be of finite thickness. This theory is relatively simple to apply, and is, moreover, formulated in such a manner that the physical behavior of the system is always readily apparent.

II. The Determination of the Dispersion Equation for a Chain of Identical Coupled Cavities

We shall consider in this paper a chain of identical loss-free symmetrical cavities coupled to each other through long slots cut in their common walls. We shall take the volume of the ith cavity of the system to be $V_i(V_i=V_{i+1}$, etc.) and shall denote by S_i' the area enclosed on the ith cavity wall by the slot or slots cut in the wall between the ith and $i-1$th cavities. Similarly, the area enclosed on the ith cavity wall by slots cut in the wall between the ith and $i+1$th cavities will be denoted by S_i''. The remaining surface area of the ith cavity will be denoted by S_i.

We shall express the fields in a cavity in terms of an expansion in the normal modes which could exist in the cavity in the absence of the slots; the coefficients of this expansion, or the excitation of such normal modes, will depend on the tangential component of electric field at the slots. The electric and magnetic fields of the nth normal mode of the ith cavity will be taken to be $\overline{E}_{i,n}$ and $\overline{H}_{i,n}$ respectively, the characteristic radian fre-

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[1] H. A. Bethe, "Theory of diffraction by small holes," *Phys. Rev.*, vol. 66, pp. 163–182; October, 1944.

[2] A. F. Stevenson, "Theory of slots in rectangular waveguides," *J. Appl. Phys.*, vol. 19, pp. 24–38; January, 1948.

[3] A. I. Akhiezer and G. Ya. Lyubarsky, "On the theory of coupled cavity resonators," *J. Tech. Phys. (USSR)*, vol. 24, pp. 1697–1708; September, 1954.

quency being $\omega_{i,n}$ with $\exp(j\omega_{i,n}t)$ dependence of all field quantities assumed.

The fields of these normal modes obey Maxwell's equations individually, and satisfy the boundary conditions at the perfectly conducting metal walls of the cavity, which are, for the ith cavity,

$$\bar{n}_i \times \overline{E}_{i,n} = 0 \quad \text{on} \quad S_i + S_i' + S_i''$$

and

$$\bar{n}_i \cdot \overline{H}_{i,n} = 0 \quad \text{on} \quad S_i + S_i' + S_i' \tag{1}$$

where $\bar{n}_i$ is the unit normal vector at the surface of the ith cavity pointing into the ith cavity.

It has been shown by Teichmann and Wigner[4] that the normal modes do not form a complete set of functions in terms of which any possible fields in the system of coupled cavities can be expressed. In the presence of coupling holes and loops, it is necessary to add irrotational components of electric and magnetic field to form the complete expansion. When only coupling slots are considered to be present, Teichmann and Wigner have shown that only an irrotational magnetic field, in addition to the normal mode fields, is required. In order to point out the principles of our method of approach, we shall first set up the complete expansion, but in the first application of the method we shall neglect the contribution of this irrotational part of the magnetic field. However, in a later part of the paper, we shall show that it is often necessary to consider this component of the field; it is, in fact, extremely important when the periodic length of the system is very much less than a slot width, since it accounts for the direct coupling from slot to slot of a multicavity chain and changes the effective slot resonant frequency.

We shall denote the irrotational component of the magnetic field in the ith cavity by $\overline{H}_{i,0}$. By definition

$$\nabla \times \overline{H}_{i,0} = 0. \tag{2}$$

Hence, $\overline{H}_{i,0}$ can be expressed as the gradient of a scalar quantity ψ_i by writing

$$\overline{H}_{i,0} = \nabla\psi_i. \tag{3}$$

Using an approach similar to that of Slater,[5] as shown in Appendix A, the fields $\overline{E}_i$ and $\overline{H}_i$ that exist in the ith cavity of the coupled system may be expressed as follows:

$$\overline{E}_i = \sum_n e_{i,n}\overline{E}_{i,n} = \sum_n \left[\frac{j\omega_{i,n}}{\omega^2 - \omega_{i,n}^2} \cdot \frac{\int_{S_i'+S_i''} (\overline{E}_i \times \overline{H}_{i,n}^*)\cdot \bar{n}_i dS}{2W_{i,n}} \overline{E}_{i,n}\right] \tag{4}$$

$$\overline{H}_i = \sum_n h_{i,n}\overline{H}_{i,n} + \nabla\psi_i = \sum_n \frac{j\omega}{\omega^2 - \omega_{i,n}^2} \cdot \left[\frac{\int_{S_i'+S_i''} (\overline{E}_i \times \overline{H}_{i,n}^*)\cdot \bar{n}_i dS}{2W_{i,n}} \overline{H}_{i,n}\right] + \nabla\psi_i, \tag{5}$$

where

$$W_{i,n} = \frac{1}{2}\int_{V_i} \epsilon E_{i,n}^2 d\tau = \frac{1}{2}\int_{V_i} \mu H_{i,n}^2 d\tau. \tag{6}$$

In the ith cavity, because $\nabla\cdot\overline{H}_{i,0} = 0$, we have the condition

$$\nabla^2\psi_i = 0 \tag{7}$$

with the boundary condition

$$\bar{n}_i \cdot \nabla\psi_i = \bar{n}_i \cdot \overline{H}_i \quad \text{on} \quad S_i' + S_i''. \tag{8}$$

On the surface of a connecting slot, the normal component of $\overline{H}$ is not zero, so that $\overline{H}_{i,0}$ exists in the ith cavity, and it is necessary to include it in the normal mode expansion.

Thus the fields in the ith cavity have been expressed entirely in terms of the tangential component of the electric field and the normal component of magnetic field at the slots which are cut in its walls.

We determine the field distribution along a slot by considering it to consist of a pair of finite-width parallel conductors with perfect shorts placed across them at planes corresponding to the ends of the slot. This is illustrated in Fig. 2. Because the slot width, or conductor spacing, is assumed to be much smaller than a wavelength, it is reasonable to assume that the electric field lines in the neighborhood of the slot always lie in a plane perpendicular to the longitudinal dimension of the slot; thus there can only be pure TE waves or TEM waves existing in the region of the slot. For a start, we shall assume that the slot fields themselves are pure TEM; this assumption may be shown to be equivalent to neglecting the presence of the irrotational mode. It is then possible to express the electrical properties of the slot in terms of transmission-line theory, the inductance and capacitance of the transmission line being computed by static considerations from the transverse dimensions of the slot. In order to write down suitable transmission-line equations to represent the excitation of a slot by the cavity fields, it is convenient to divide the currents flowing in the neighborhood of the slot into two parts. The first part is the current, I, which flows around the slot or along the length of the transmission line; this current is directly associated with the component of magnetic field normal to the plane of the slot. The second part is the exciting current, j_y, which flows on the surface of a cavity into the slot in a direction perpendicular to the longitudinal dimension of the slot; this current is associated with the component of magnetic field in the cavities which lies along the longi-

[4] T. Teichmann and E. P. Wigner, "Electromagnetic field expansions in loss-free cavities through holes," *J. Appl. Phys.*, vol. 24, pp. 262–267; March, 1953.

[5] J. C. Slater, "Microwave Electronics," D. Van Nostrand Co., Inc., New York, N. Y.; 1950.

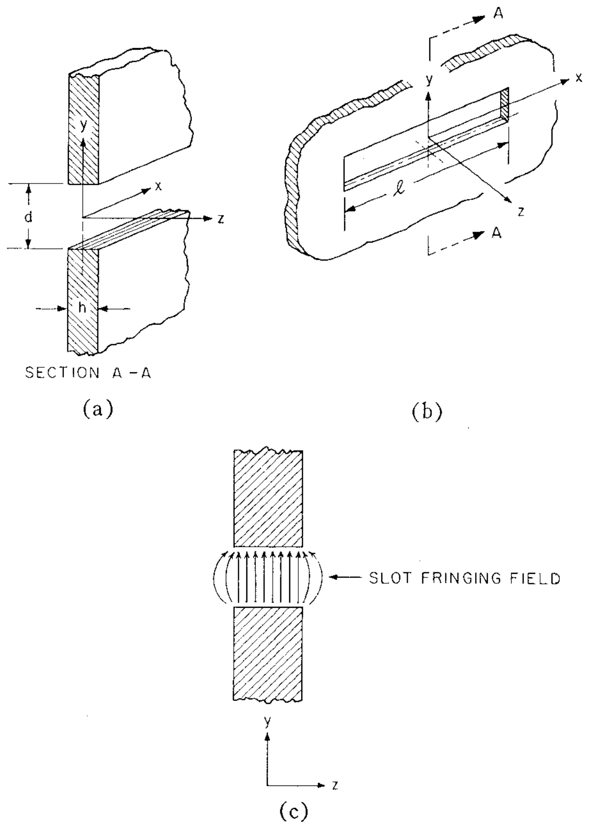

Fig. 2—(a) and (b) Views of slot with coordinate system shown. (c) Section through slot showing behavior of the electric field.

tudinal dimension of the slot. For a slot which is cut in the common wall of the $i-1$th and ith cavities, the current j_y flowing into the slot is given in value by the expression

$$j_y = j_{i-1,y}'' + j_{i,y}' = \bar{n}_i \times (\overline{H}_i + \overline{H}_{i-1})_x \text{ on } S_i' + S_{i-1}'' \quad (9)$$

where the subscript y represents the direction along the surface of a cavity, perpendicular to the longitudinal direction of a slot and j_y denotes the *total* current flowing into the slot on both sides of the wall between the $i-1$th and ith cavities. The significance of these two current components is illustrated in the transmission-line representation in Fig. 3. This representation yields two differential equations for the voltage and current distributions along a slot in terms of the fields at a slot; these may be written as follows:

$$\frac{\partial \phi}{\partial x} = -j\omega LI \quad (10)$$

$$\frac{\partial I}{\partial x} - j_y = -j\omega C\phi \quad (11)$$

with the boundary conditions

$$\phi = 0 \quad \text{at} \quad x = \pm \frac{l}{2}, \quad (12)$$

where C is the capacitance per unit length of the slot, L is the inductance per unit length of the slot, ϕ the volt-

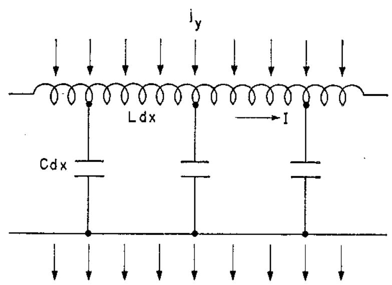

Fig. 3—Transmission-line representation of the slot with net current flow across the slot.

age across a slot, l the length of a slot, and x denotes distance along the slot from its center. Eqs. (10) and (11) yield the differential equation for the voltage distribution across a slot, in terms of the driving currents due to the cavity modes,

$$\frac{\partial^2 \phi}{\partial x^2} + k^2\phi = -jkZ_0 j_y \quad (13)$$

where $k=\omega/c$ and $Z_0=\sqrt{L/C}$, the characteristic impedance of the slot. Writing this equation as

$$\frac{\partial^2 \phi}{\partial x^2} + k^2\phi = -f(x), \quad (14)$$

the solution (see, for example, Webster[6]) may be expressed as follows:

$$\phi = \frac{\sin k(l/2 - x)}{k \sin kl} \int_{-l/2}^{x} \sin k(l/2 + \xi) f(l/2 + \xi) d\xi + \frac{\sin k(x + l/2)}{k \sin kl} \int_{x}^{l/2} \sin k(l/2 - \xi) f(l/2 - \xi) d\xi. \quad (15)$$

By writing

$$\phi = \int_0^d E_y dy, \quad (16)$$

we thus have an expression for the tangential component of the electric field at the slot in terms of the cavity fields in the neighborhood of a slot of width d. The characteristic impedance Z_0 can be determined analytically by a conformal transformation or experimentally in an electrolytic tank. Thus finite connecting wall thickness may be taken into account. For the limiting case of zero wall thickness the inverse cosine transformation may be used.[7]

We have in (4) and (5) an expression for the fields in the ith cavity in terms of the tangential component of the electric field at (*i.e.*, the voltage across) the slots. The current which flows into the region of a slot is expressible in terms of these fields and hence in terms of the voltages across the slots. This current is the current

[6] A. G. Webster, "Partial Differential Equations of Mathematical; Physics," Dover Publications, New York, N. Y.; 1955.

[7] S. Ramo and J. R. Whinnery, "Fields and Waves in Modern Radio," John Wiley and Sons, Inc., New York, N. Y., 2nd ed., pp. 135–138; 1953.

which excites the slot fields, and in terms of which the voltage across a slot was expressed in (13) and (15). As a result, we have an expression for the current flowing into the region of a slot in terms of the voltage across a slot, and an expression for the voltage across a slot in terms of the current flowing into the slot region. On combining these expressions, a single integral equation whose eigenvalues give the dispersion equation of the coupled system may be obtained. For simplicity, we shall only consider a system made up of identical cavities coupled by a single slot in each cavity wall; the extrapolation of the method to a multislot coupling system may be readily accomplished.

Following the procedure outlined above, we first write down the current j_y in terms of the normal mode fields in the ith and $i-1$th cavities. Thus from (5) and (9) it follows that

$$j_y = \bar{n}_i \times \Big[\sum_n (h_{i,n}\overline{H}_{i,n})_{S_i'} + \sum_n (h_{i-1,n}\overline{H}_{i-1,n})_{S_{i-1}''}\Big]_y \quad (17)$$

where, at this juncture, we have neglected the contribution of the irrotational field. A further transformation of (17), by writing it in the form

$$j_y = \sum_n (h_{i,n} j_{i,n,y}') + \sum_n (h_{i-1,n} j_{i-1,n,y}'') \quad (18)$$

is convenient, where $j_{i,n}'$ and $j_{i-1,n}''$ are the surface currents, in the region of the slot of the nth normal modes in the ith and $i-1$th cavities respectively. In terms of the normal mode fields

$$j_{i,n}' = (\bar{n}_i \times \overline{H}_{i,n})_{S_{i,n}'}$$
$$j_{i-1,n}'' = (\bar{n}_i \times \overline{H}_{i-1,n})_{S_{i-1,n}''}. \quad (19)$$

The form of these equations may now be simplified by taking the nth normal mode fields in each cavity to be identical, so that we can use Floquet's theorem to write

$$h_{i-1,n} = h_{i,n} e^{j\beta L} \quad (20)$$

where L is the periodic length of the system, and βL the phase delay between cavities.

Moreover, as the cavities are symmetrical and identical,

$$j_{i,n}'' = j_{i-1,n}'' = \pm j_{i,n}' \quad (21)$$

where the choice of sign depends on the form of the nth mode of the cavity. For example, the negative sign is appropriate for the TM_{010} mode of a cylindrical cavity, the positive sign for the TM_{011} mode of the same cavity. Thus, we can simplify (18) by the use of (20) and (21) and write

$$j_y = \sum_n h_{i,n}(1 \pm e^{j\beta L}) j_{i,n,y}'. \quad (22)$$

In order to determine the final dispersion relation, we must first find $h_{i,n}$ in terms of j_y. To do this, we may make use of (5) and (19) to write

$$h_{i,n} = \frac{-j\omega}{\omega^2 - \omega_{i,n}^2} \frac{\int_{S_i''} \overline{E}_i \cdot j_{i,n}'' dS + \int_{S_i'} \overline{E}_i \cdot j_{i,n}' dS}{2W_{i,n}}. \quad (23)$$

We may use Floquet's theorem and (21) to write (23) in the form:

$$h_{i,n} = \frac{-j\omega}{\omega^2 - \omega_{i,n}^2} \frac{\int_{S_i''} \overline{E}_i \cdot j_{i,n}'(1 \pm e^{-j\beta L}) dS}{2W_{i,n}}. \quad (24)$$

In order to express (24) in terms of the voltage across the slot, it is convenient to make the assumption that $j_{i,n}'$ is constant over the width of the slot, although this assumption is not strictly necessary. With this assumption, however, we may carry out the integration over the width of the slot and write

$$h_{i,n} = \frac{-j\omega}{\omega^2 - \omega_{i,n}^2} \frac{(1 \pm e^{-j\beta L}) \int_{-l/2}^{l/2} \phi_i j_{i,n,y}' dx}{2W_{i,n}}. \quad (25)$$

We now find j_y in terms of ϕ by substituting (25) in (22) to yield the following expression:

$$j_y = \sum_n \frac{-j\omega(1 \pm \cos \beta L) j_{i,n,y}' \int_{-l/2}^{l/2} \phi_i j_{i,n,y}' dx}{(\omega^2 - \omega_{i,n}^2) W_{i,n}}. \quad (26)$$

It will be recalled that we have already found ϕ_i in terms of j_y in (15); thus we are in a position to eliminate ϕ_i from (15) and (26) and to write the following integral equation for j_y:

$$j_y = \sum_n \frac{-j\omega(1 \pm \cos\beta L) j_{i,n,y}'}{(\omega^2 - \omega_{i,n}^2) W_{i,n}} \int_{-l/2}^{l/2} dx \cdot \Big\{ j_{i,n,y} \Big[\frac{\sin k(l/2 - x)}{k \sin kl} \int_{-l/2}^{x} \sin k(l/2 + x) f(l/2 + \xi) d\xi + \frac{\sin k(x + l/2)}{k \sin kl} \int_{x}^{l/2} \sin k(l/2 - \xi) f(l/2 - \xi) d\xi \Big] \Big\} \quad (27)$$

where

$$f(x) = -jkZ_0 j_y(x).$$

Eq. (27) is a Fredholm integral equation of the second kind which may in principle be solved to give the possible modes of the coupled system. Such a procedure is, in practice, too difficult, so that it is necessary to make certain approximations of a type which will be described in the next section. Alternatively, it is sometimes convenient and more accurate to set up the solution for this coupled system in a variational form. The procedure for obtaining this variational form of the solution is given in Appendix B.

III. Application to a Particular Cavity Chain

The particular resonators which we shall study for the purpose of illustrating our methods will be cylindrical cavities coupled to one another by long, circumferential, narrow, identical slots cut in the common walls of adjacent cavities, as shown in Fig. 1.

The dominant cavity mode in the system, before the slots are cut in the disks, will be taken to be the TM_{010} mode. Its characteristic radian frequency will be denoted by ω_1.

We have developed an expression, (27), for the characteristic frequencies of a coupled system. We shall make use of an approximation to this expression which employs only the dominant (TM_{010}) mode in the series expansion. This gives rise to a simplified coupling equation. At a later stage, the effect of the irrotational magnetic mode introduced earlier will be considered.

A. Dominant Mode Expansion

The circumferential coupling slot will be taken to be cut close to the cylindrical walls. The slot length and thickness are denoted by l and h, respectively, and the cavity radius and length by a and b, respectively. Thus, the periodic length of the system is $L=a+h$. The dominant TM_{010} mode of the cavities has the following fields:

$$E_z = E_0 J_0(k_1 r)$$
$$H_\phi = \frac{jE_0}{\eta} J_1(k_1 r) \tag{28}$$

where $J_\nu(x)$ is a Bessel function of the first kind and νth order; and

$$k_1 = \frac{p_{01}}{a} = \frac{2.405}{a},$$

p_{01} being the first zero of the zero-order Bessel function. We shall refer to this dominant mode by the subscript 1. In the normal mode expansion we shall only use this dominant mode term to determine the coupling. The fields in the cavities are therefore expressed as follows:

$$\overline{E}_i = e_{i,1}\overline{E}_{i,1} + \overline{E}_i'$$
$$\overline{H}_i = h_{i,1}\overline{H}_{i,1} + \overline{H}_i', \tag{29}$$

where $\overline{E}_i'$ and $\overline{H}_i'$ are the remaining fields orthogonal to the dominant mode. The current j_y which drives the slot is assumed to arise only from the dominant mode fields. These fields have no θ variation, which, for circumferential slots, means that they have no x variation. With the current being independent of the x coordinate, (15) gives ϕ_i in the form

$$\phi_i = -\frac{jZ_0 j_y}{k}\left(1 - \frac{\cos kx}{\cos kl/2}\right). \tag{30}$$

By retaining only the dominant mode in (27), and noting that j_y is now independent of the x coordinate, we obtain the following dispersion relation:

$$\frac{\Omega_1(1-\Omega_1^2)}{\tan\left(\frac{\rho\Omega_1\pi}{2}\right) - \frac{\rho\Omega_1\pi}{2}} = \alpha_\pi \sin^2\frac{\beta L}{2} = \alpha \tag{31}$$

where

$$\alpha_\pi = \frac{4j_1^2 Z_0 c^2}{W_1\omega_1^2}; \tag{32}$$

and

$$\Omega_1 = \frac{\omega}{\omega_1}; \qquad \rho = \frac{\omega_1}{\omega_{s1}}; \qquad \frac{\omega_{s1} l}{2c} = \frac{\pi}{2}.$$

Since the cavities are identical we now omit the subscript referring to the particular cavity.

Eq. (31) yields on solution an infinite number of values of ω for each value of α, the dimensionless coupling coefficient. For $\alpha_\pi \sin^2 \beta L/2=0$ (for this system, when $\beta L=0$) these values of ω are ω_1, ω_{s1}, ω_{s2}, ω_{s3}, $\cdots$, corresponding to the poles of the denominator and zero of the numerator of (31). Solving the equation as βL increases in value from zero to π, an infinite number of pass bands results, one associated with the cavity dominant mode characteristic frequency ω_1, and the others associated with the frequency ω_{sp} which makes the slot p half-wavelengths long. We shall concern ourselves only with the two lowest frequency pass bands: the one associated with ω_1, and the other associated with ω_{s1}. Plotted on an ω-β diagram, the higher frequency curve has a positive slope $(d\omega/d\beta>0)$ and the low frequency curve has a negative slope $(d\omega/d\beta<0)$.

Numerical solutions of (31) are given, with the quantity Ω_1 plotted as a function of ρ, from $\rho=0.5$ to $\rho=2.5$ for several values of α from zero to unity in Fig. 4. With α known, the ω-β diagram may be plotted for the coupled-cavity system by taking values from Fig. 4, using the appropriate value of ω_1/ω_{s1}.

In order to determine the dispersion relation for a particular system, we must evaluate the coupling coefficient. In the dominant mode we have

$$i_{1y}^2 = \frac{E_1^2(0)}{\eta^2} J_1^2(k_1 a), \tag{33}$$

assuming the slot is cut near the maximum radius where the magnetic field is a maximum.

Since the dominant mode has no θ variation, the energy stored in this mode, W_1, is expressed by a single integral,

$$W_1 = b\int_0^a \frac{\epsilon\,|E_z|^2 2\pi r dr}{2}. \tag{34}$$

On substituting (33) and (34) into (32) we obtain

$$\alpha_\pi = \frac{8Z_0c^2}{\eta^2\pi\epsilon b a^2\omega_1^3}. \tag{35}$$

For the TM_{010} mode $\lambda_1=2.61a$; hence

$$\omega_1{}^3 = \frac{8\pi^3 c^3}{17.8 a^3}, \tag{36}$$

and since $\epsilon = 1/\eta c$ where η is the characteristic impedance of free space, we obtain finally a simple formula for the coupling coefficient

$$\alpha_\pi = 0.18\left(\frac{a}{b}\right)\left(\frac{Z_0}{\eta}\right). \tag{37}$$

This coupling coefficient, as we might expect, depends upon the static properties of the slot, Z_0, and the ratio of the cavity radius to the cavity length, a/b.

The dispersion characteristics for a typical structure are shown in Fig. 5. Agreement between theory and experiment is seen to be good in the cavity pass band. In the slot pass band some disagreement is observed, and in structures where the slot width is comparable to the cavity length ($d \approx b$ in Fig. 1) this disagreement becomes large. This is a result of the exclusion of the irrotational component of the magnetic field from the field expansion.

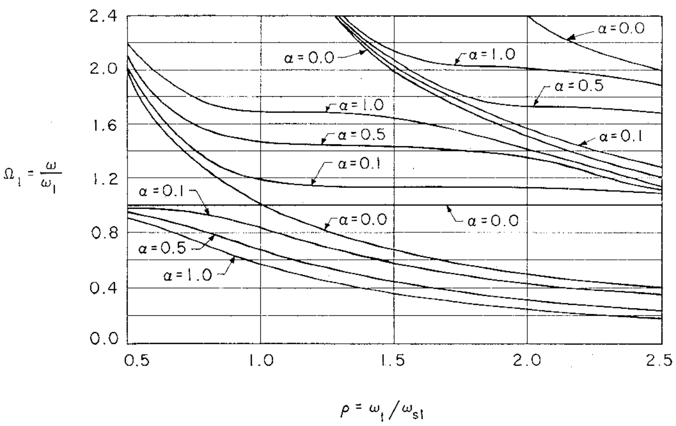

Fig. 4—Solutions to the dispersion relation (31).

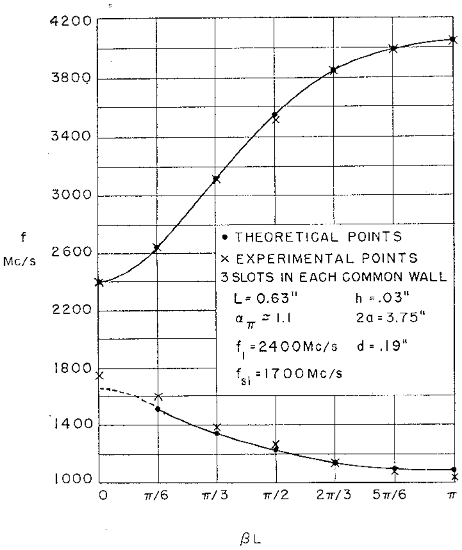

Fig. 5—Typical Brillouin diagram between $\beta L = 0$ and $\beta L = \pi$ for the two lowest pass bands for a coupled-cavity system of the type shown in Fig. 1.

B. Inclusion of the Irrotational Mode

We shall now consider the way in which the irrotational magnetic mode arises in the coupled system under study. It was shown that the fields in the slots gave rise to voltages and currents which obey the normal transmission-line equations with an additional driving current term. The current which flows along the slot is directly related to the normal component of magnetic field at the slot. If this slot were cut in an infinite plane, it could be assumed that the slot fields were pure TEM. However, because the slot is actually cut in the wall of a closed resonator, the magnetic field lines must be continuous through the slot and must be closed, as shown in Fig. 6. Therefore, a longitudinal component of the magnetic field must exist which is associated with the normal magnetic field at the slot. Mathematically, this longitudinal field component can only be expressed by the irrotational field.

This longitudinal component of magnetic field, which is represented by the irrotational mode, has two effects: 1) It gives the effect of shunt inductance across the slot and, hence, changes the resonant frequency of the slot. 2) It introduces a component of field which can directly provide large amounts of slot-slot coupling or mutual inductance between the slots. In order to illustrate what occurs, we consider the system of slot-coupled resonators shown in Fig. 7. There will be an irrotational component of magnetic field originating from the currents which flow along slot A. Associated with this irrotational magnetic field there will be currents which flow in the walls of the two cavities perpendicular to the slot A. The wall current which flows

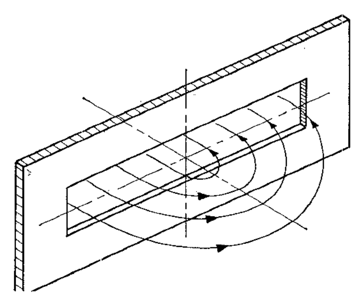

Fig. 6—Sketch of magnetic field lines associated with the normal component of the magnetic field at the slot.

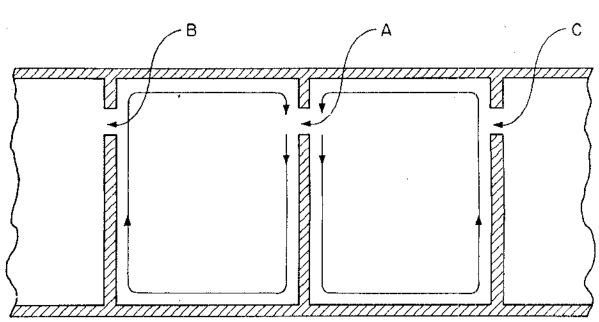

Fig. 7—Current associated with the longitudinal component of the irrotational part of the magnetic field at the slot.

across slot A we call the "self" current; it may be represented by the current flowing in a shunt inductance across slot A, and it causes an increase in the resonant frequency of the slot. Similar currents arise from the slots B and C adjacent to slot A in Fig. 7. A portion of these currents flow into slot A as "mutual" currents, giving slot-slot coupling which may be represented as mutual inductance between the adjacent slots A-B and A-C. If there is zero phase shift between adjacent slots, the mutual current from the two slots adjacent to a given slot and the self-current of the slot will tend to cancel. For a $\pi/2$ phase shift there will be no net effect from the mutual currents arising from adjacent slots, and at π phase shift the self and mutual currents will add. The over-all effect is that the bandwidth of the slot band is reduced below the value predicted by a theory which does not include the irrotational component of the magnetic field. The effect of this extra component of field is less important in the cavity pass band than in the slot pass band.

The situation which has been described is that which is the case for identically placed slots between the cavities. If the slots at one plane are rotated with respect to the slots at adjacent planes the effect of the mutual currents is reduced.

Fig. 8 gives experimental results in a system with the slots aligned and in a system with the slot rotated 90° to adjacent slots. It is noted that with the mutual effect eliminated, the self-effect raises the slot band in frequency but does not reduce its total bandwidth.

The behavior of this irrotational mode depends on the determination of the function which is the solution to the Neumann problem in a cylindrical cavity resonator with the given boundary conditions. With the problem solved, the gradient of this function is taken along the slot itself and along adjacent slots to give the self- and mutual currents, respectively. The solving of the Neumann problem involves dealing with slowly converging series of trigonometric and Bessel functions. The convergence of these series depends critically on the fact that the width of the slot is finite. It has not been possible to sum these series in closed form; consequently only a semiquantitative picture will be given of the effect of the irrotational mode, or the "slot-slot" coupling which it represents. The slot is of width d, and is placed at $z=L$, as shown in Fig. 9. We have for a given cavity

$$\nabla^2\psi = 0$$

$$\frac{\partial\psi}{\partial n} = H_n \quad \text{on} \quad S' + S''$$

$$\frac{\partial\psi}{\partial n} = 0 \quad \text{on} \quad S. \tag{38}$$

The most general solution for ψ which obeys Laplace's equation and the boundary conditions of the cylindrical cavity is

$$\psi = \sum_{mn} A_{mn} \cosh(\beta_n z) J_m(\beta_n r) \sin(m\theta) \tag{39}$$

with

$$J_m'(\beta_n a) = 0.$$

The current along the slot, I, is related to the normal magnetic field H_n at the slot as follows:

$$LI = \mu H_n d \tag{40}$$

where L is the inductance per unit length of the slot, as defined previously. The field is assumed to be constant across the width of the slot. From (10), (11), and (30) we have

$$I = I_0 \sin(ak\theta), \tag{41}$$

where I_0 is the maximum value of the total current flowing along the slot. We note that $L/\mu = Z_0/\eta$ where Z_0 and η are the characteristic impedances of the slot and free space, respectively. Thus, from (40), at the slot we have

$$\left.\frac{\partial\psi}{\partial z}\right|_{z=L} = \frac{Z_0}{\eta}\frac{I_0}{d}\sin(ak\theta). \tag{42}$$

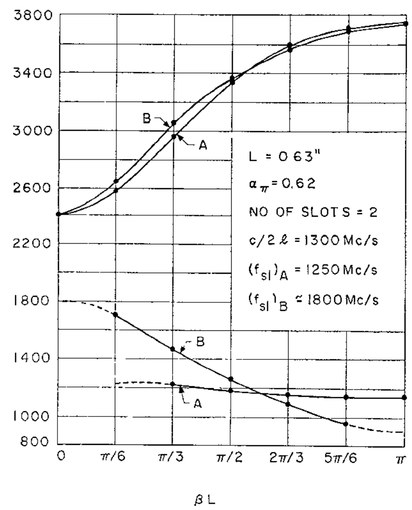

Fig. 8—Brillouin diagrams for coupled-cavity system. A. Adjacent slots in line. B. Adjacent slots rotated 90°.

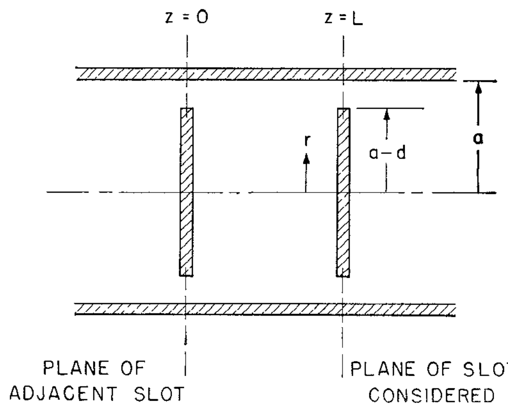

Fig. 9—Coordinate system for calculation of irrotational component of the magnetic field.

Eqs. (38), (39), and (42) may be combined[8] to give an approximate expression for the irrotational field in the cavities and the self- and mutual currents associated with this field give an added current flowing in the slot region at the cavity

$$j_{y0} = -P\frac{\partial I}{\partial x} + Q\frac{\partial I}{\partial x}\cos(\beta L), \tag{43}$$

where

$$P = \frac{Z_0}{\eta}\sum_n \frac{1}{\tan(\beta_n L)}\,\frac{\beta_n a}{(\beta_n a)^2 - 1}\left(\frac{\sin(\beta_n d/2)}{\beta_n d/2}\right)^2$$

$$Q = \frac{Z_0}{\eta}\sum_n \frac{1}{\sinh(\beta_n L)}\,\frac{\beta_n a}{(\beta_n a)^2 - 1}\left(\frac{\sin(\beta_n d/2)}{\beta_n d/2}\right)^2. \tag{44}$$

We have made the approximation of using only the first θ term in the double summation of (39). The strongly converging terms

$$\left(\frac{\sin(\beta_n d/2)}{\beta_n d/2}\right)^2$$

arise from integrations over the finite width of the slots. The net current driving the slot is $j_y + j_{y0}$. Modifying (11) accordingly, (31) now becomes

$$\frac{\Omega_1(1-\Omega_1^2)}{\tan(\rho'\Omega_1\pi/2) - \rho'\Omega_1\pi/2} = \alpha\sqrt{1+P-Q\cos\beta L}\,\sin^2(\beta L/2) \tag{45}$$

where

$$\rho' = \frac{\rho}{\sqrt{1+P-Q\cos\beta L}}\,.$$

Thus, we have a coupling equation which is of the same form as the coupling equation obtained without the inclusion of the irrotational mode, but with different slot resonant frequencies and coupling coefficients,

$$\omega_{s1}' = \omega_{s1}\sqrt{1+P-Q\cos\beta L}$$
$$\alpha_\pi' = \alpha_\pi\sqrt{1+P-Q\cos\beta L}. \tag{46}$$

The quantity P, the self-term, will in general be larger than the quantity Q, the mutual term, from adjacent slots. The slot resonant frequency will be higher than in the case without the inclusion of the irrotational mode at $\beta L = 0$. As βL increases, the slot resonant frequency will increase in value. Based on this higher slot resonant frequency, the coupling between cavities will, for certain βL, give a characteristic frequency below this resonant frequency, as is given by the equation whose solutions are given in Fig. 4. The net result will be a decrease in the total bandwidth of the pass band associated with the slot resonant frequency. The sum of the first ten terms of the expansion, in (44), gives a good approximation of the expansions. In (45) the values of P and Q thus obtained account for the departure from theory in the lower pass band of Fig. 5.

IV. Determination of Field Distributions

In assessing the usefulness of microwave filters for interaction with an electron beam, the quantity $E^2(0)/W$ is a figure of merit where $E^2(0)$ is the square of the longitudinal component of the electric field on the axis and W is the energy stored per periodic length. This quantity will be considered.

It is possible to make accurate estimates of the ω-β curves for large coupling between cavities by using a theory based on only the dominant mode in the normal mode expansions of the fields. Although the ω-β curves can be accurate at frequencies far from the dominant mode resonance, the dominant mode alone is not sufficient to express the values of the fields in the cavities. This is illustrated by the fact that, in Appendix B, we found an expression for ω in a variational form; thus, by using only a rough estimate of the fields in the slot we obtain a value for the frequency which is less in error than the estimate of the fields themselves.

A theorem exists for lossless periodic transmission system which states that the time-average electric stored energy per period is equal to the time-average magnetic stored energy per period in the pass band. A proof of this theorem has been given by several authors.[9] Using this theorem we are able to express the stored energy per period in terms of only the electric energy.

The cylindrical cavity resonators with which we are concerned have lengths much smaller than their diameters. The characteristic frequencies of the normal modes having variations of the field in the direction of the cavity length (z-varying modes) will thus be very much above the characteristic frequencies of the corresponding non-z-varying modes, and will have much smaller amplitudes in the expansion. Therefore, the electric energy stored in the coupled system under consideration, except in and around the region of the slot, comes largely from the dominant mode and other non-θ-, non-z-varying modes. The dominant mode has a longitudinal component of the electric field which has a maximum value on the axis and falls to zero at the circumference of the cavity. Higher-order modes have further zeros between the maximum value on the axis and zero at the circumference. In the coupled system, for values of frequency above the dominant mode frequency, since one or more additional zeros in the longitudinal component of the electric field must occur, the dominant mode alone cannot give a good representation of the field. This is illustrated in Fig. 10. Thus the stored energy in the cavities must have appreciable contributions from several modes.

[8] M. A. Allen, "Coupling of Multiple Cavity Systems," Microwave Lab., W. W. Hansen Labs. of Physics, Stanford University, Stanford, Calif., M.L. Rept. No. 584; 1959.

[9] D. A. Watkins, "Topics in Electromagnetic Theory," John Wiley and Sons, Inc., New York, N. Y.; 1958.

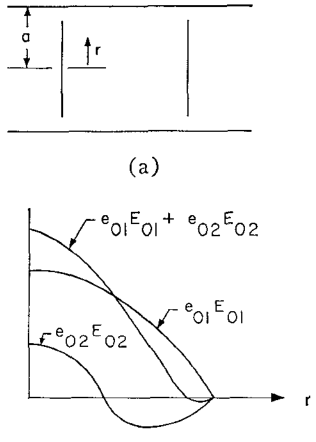

Fig. 10—For structure in (a), the plot of the amplitudes as a function of radial distance for the first two terms of the normal mode expansions is given in (b).

We now consider the slot region. If the total field $\overline{E}$ everywhere is divided into two parts

$$\overline{E} = \left(\sum_n e_{0n}\overline{E}_{0n} \right) + \overline{E}' \tag{47}$$

where the subscript $0n$ refers to the TM_{0n0} modes, then a large proportion of the electric energy stored in the slot region would arise from the part of the field represented by $\overline{E}'$. Because of the orthogonality of the normal modes, the stored energy W per periodic length is given by

$$\begin{aligned} W &= 1/2 \int_{V_i} \epsilon \overline{E}\cdot\overline{E}^* d\tau \\ &= \sum_n e_{0n}{}^2 W_{0n} + 1/2 \int_{V_i} \epsilon \overline{E}'\cdot\overline{E}'^* d\tau. \end{aligned} \tag{48}$$

The summation in (48) may be summed in closed form.[8] The volume integral on the right-hand side of (48) has most of its contribution from the part of the volume near the slot. The behavior of the field in the slot region was derived from considerations of the excitations of a TEM wave guided by parallel-plane conductors leading to a static description of the fields there. The energy in the fields at the slot is as would be derived for a static distribution of the fields. Thus, the time-average electric energy stored by a slot W_{sE} is given by

$$\begin{aligned} W_{sE} &= 1/2 \int_{V_i} \epsilon \overline{E}'\cdot\overline{E}'^* d\tau \\ &= 1/2 \int_{-l/2}^{l/2} C\phi^2 dx. \end{aligned} \tag{49}$$

We assume that (49) includes all the energy stored outside the TM_{0n0} modes in the coupled system, and so avoid having to consider the z-varying and θ-varying modes' contributions separately. The voltage ϕ may therefore be obtained from (30).

If now the field on the axis is considered, then, of the normal modes which have a longitudinal component of electric field on the axis, the non-z-varying modes will, as in the considerations of energy, contribute more to the normal mode expansion for the electric field on the axis than the z-varying modes. Thus, for cavities of large diameter-to-length ratio, we express the electric field on the axis as an expansion based only on the TM_{0n0} normal modes,

$$\overline{E}(0) = \sum e_{0n}\overline{E}_{0n}(0). \tag{50}$$

We note that the azimuthal-varying modes have no longitudinal component of the electric field on the axis, and they need not be considered as they are in computing W, the total stored energy per period. Using values of frequency based on the dominant mode expansion, (50) may be summed in a closed form.[8]

For the type of coupling we are considering, the irrotational mode makes no direct contribution to the electric energy stored per period; it contributes only to the stored magnetic energy. However, this irrotational mode does have an effect on the values of the coefficients of the normal mode expansions. Values of frequency based on this irrotational mode should be included in the determination of the quantity $E^2(0)/W$, especially in the slot band.

Some typical theoretical and experimental results are shown in Fig. 11. If only the dominant mode had been used in the normal mode expressions the large values of $E^2(0)/W$ at large values of βL in the upper pass band would not have been predicted.

Appendix A

Evaluation of the Coefficients of the Normal Mode Expansions of the Fields in a Cavity

The normal modes of the ith cavity are given by

$$\overline{E}_{i,n}(\bar{r}), \qquad \overline{H}_{i,n}(\bar{r}).$$

$\omega_n^{(i)}$ are the associated characteristic frequencies, with

$$e^{-j\omega_n^{(i)}t}$$

time dependence assumed. Maxwell's equations with electric boundary conditions are obeyed:

$$\left.\begin{aligned} \bar{\nabla} \times \overline{E}_{i,n} &= -j\omega_{i,n}\mu H_{i,n} \\ \bar{\nabla} \times \overline{H}_{i,n} &= j\omega_{i,n}\epsilon\overline{E}_{i,n} \\ \bar{\nabla} \times \overline{E}^*_{i,n} &= j\omega_{i,n}\mu\overline{H}^*_{i,n} \\ \bar{\nabla} \times \overline{H}^*_{i,n} &= -j\omega_{i,n}\epsilon\overline{E}^*_{i,n} \end{aligned}\right\} \tag{51}$$

with

$$\bar{n} \times \overline{E}_{i,n} = 0 \quad \text{on} \quad S_i + S_i' + S_i''. \tag{52}$$

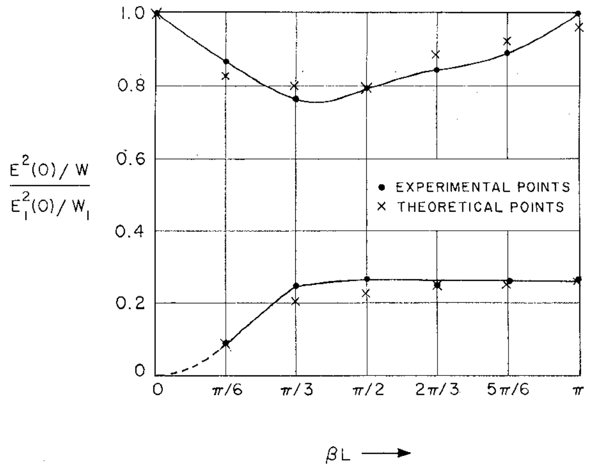

Fig. 11—Normalized $E^2(0)/W$ plotted as a function of βL for the two lower pass bands of a coupled-cavity system of the type shown in Fig. 1.

The required fields $\bar{E}(\bar{r})$ and $\bar{H}_i(\bar{r})$ satisfy Maxwell's equations with frequency ω in the coupled system:

$$\left.\begin{aligned} \bar{\nabla} \times \bar{E}_i &= -j\omega\mu\bar{H}_i \\ \bar{\nabla} \times \bar{H}_i &= j\omega\epsilon\bar{E}_i \\ \bar{\nabla} \times \bar{E}_i^* &= j\omega\mu\bar{H}_i^* \\ \bar{\nabla} \times \bar{H}_i^* &= j\omega\epsilon\bar{E}_i^* \end{aligned}\right\} \tag{53}$$

with

$$\bar{n} \times \bar{E}_i(r) = 0 \quad \text{on} \quad S_i - S_i' - S_i''. \tag{54}$$

We consider the following:

$$\int_V \bar{\nabla}\cdot(\bar{E}_i \times \bar{H}^*_{i,n})d\tau$$

$$= \int_{V_i} [\bar{H}^*_{i,n}\cdot\bar{\nabla}\times\bar{E}_i - \bar{E}_i\cdot\bar{\nabla}\times\bar{H}^*_{i,n}]d\tau$$

$$= \int_{V_i} [-j\omega\mu\bar{H}_i\cdot\bar{H}^*_{i,n} + j\omega_{i,n}\epsilon\bar{E}\cdot\bar{E}^*_{i,n}]d\tau. \tag{55}$$

By using (51) and (53) and employing Gauss's theorem we have

$$\omega_{i,n}\int_{V_i}\epsilon(\bar{E}_i\cdot\bar{E}^*_{i,n})d\tau - \omega\int_{V_i}\mu(\bar{H}^*_{i,n}\cdot\bar{H}_i)d\tau = \frac{1}{j}\int_{S_i}(\bar{E}\times\bar{H}^*_{i,n})\cdot n dS; \tag{56}$$

and, similarly, by considering

$$\int_{Vi}\bar{\nabla}\cdot(\bar{E}^*_{i,n}\times\bar{H}_i)d\tau$$

and (52) we can show that

$$\omega\int_{V_i}\epsilon(\bar{E}^*_{i,n}\cdot\bar{E})d\tau - \omega_{i,n}\int\mu(\bar{H}^*_{i,n}\cdot\bar{H}_i)d\tau = 0. \tag{57}$$

The normal mode expansions are

$$\bar{E}_i = \sum e_{i,n}\bar{E}_{i,n} \tag{58}$$

$$\bar{H}_i = \sum h_{i,n}\bar{H}_{i,n} + \bar{\nabla}\psi_i \tag{59}$$

where

$$e_{i,n} = \frac{\int_{V_i}\epsilon_n E_i\cdot\bar{E}_{i,n}{}^* d\tau}{\int_{V_i}\mu\bar{E}_{i,n}\cdot\bar{E}_{,in}d\tau} \tag{60}$$

and

$$h_{i,n} = \frac{\int_{V_i}\mu\bar{H}_i\cdot\bar{H}_{i,n}d\tau}{\int_{V_i}\mu\bar{H}_{i,n}\cdot\bar{H}_{i,n}d\tau}. \tag{61}$$

By using (56) and (57) we obtain

$$e_{i,n} = \frac{-j\omega_{i,n}}{\omega^2 - \omega_{i,n}{}^2}\,\frac{\int_{S_i'+S_i''}(\bar{E}_i\times\bar{H}^*_{i,n})\cdot\bar{n}_i dS}{2W_n{}^{(i)}} \tag{62}$$

$$h_{i,n} = \frac{j\omega}{\omega^2 - \omega_{i,n}{}^2}\,\frac{\int_{S_i'+S_i''}(\bar{E}_i\times\bar{H}_{i,n}{}^*)\cdot\bar{n}_i dS}{2W_{i,n}} \tag{63}$$

since the surface integrals only have values at the slot surfaces.

Appendix B

Variational Form of the Solution

We shall start with the basic equations, (13) and (26), for the excitation of the slots. These equations combine to give

$$\frac{\partial^2\phi_i}{\partial x^2} + k^2\phi_i$$

$$= \frac{kZ_0\sum_n \frac{\omega(1\pm\cos\beta L)}{\omega^2-\omega_{i,n}{}^2} j_{i,n,y}'\int_{-l/2}^{l/2} j^*_{i,n,y}{}'\phi_i dx}{2W_{i,n}} \tag{64}$$

for the case of a chain of identical cavities. Rewriting, we obtain

$$\frac{\partial^2\phi_i}{\partial x^2} + \frac{\omega^2}{c^2}\phi_i$$

$$= \frac{Z_0}{c}\sum_n \frac{(1\pm\cos\beta L)\omega^2}{(\omega^2-\omega_{i,n}{}^2)}\,\frac{j_{i,n,y}{}'}{2W_{i,n}}\int_{-l/2}^{l/2} j^*_{i,n,y}\phi_i dx. \tag{65}$$

Multiplying (65) by ϕ_i^*, and integrating once along the slot, we obtain

$$\int_{-l/2}^{l/2}\left(\frac{\partial^2\phi_i}{\partial x^2}+\frac{\omega^2}{c^2}\phi_i\right)\phi_i^*\,dx=\frac{Z_0}{c}$$

$$\cdot\sum_n \frac{\omega^2(1\pm\cos\beta L)}{2(\omega^2-\omega_{i,n}^2)W_{i,n}}\int_{-l/2}^{l/2}\bar{j}_{i,n,y}\phi_i^*dx\int_{-l/2}^{l/2}\bar{j}_{i,n,y}^*\phi_i dx. \quad (66)$$

Integrating by parts the second term on the left-hand side of (66) we have

$$\frac{\omega^2}{c^2}\int_{-l/2}^{l/2}\phi_i\phi_i^*dx-\int_{-l/2}^{l/2}\frac{\partial\phi_i}{\partial x}\frac{\partial\phi_i^*}{\partial x}=\frac{Z_0}{c}$$

$$\cdot\sum_n \frac{\omega^2(1\pm\cos\beta L)}{(\omega^2-\omega_{i,n}^2)2W_{i,n}}\int_{-l/2}^{l/2}\bar{j}_{i,n,y}\phi_i^*dx\int_{-l/2}^{l/2}\bar{j}_{i,n,y}^*\phi_i dx \quad (67)$$

since

$$\phi(\pm l/2)=0$$

Let us assume that $\phi_{0,i}$ and ω_0 solve (67). A trial function for the voltage is expressed as follows:

$$\phi_i(x)=\phi_{0,i}(x)+\epsilon_i(x). \quad (68)$$

Since the values of the voltage at the ends of the slot are known,

$$\epsilon_i(\pm l/2)=0. \quad (69)$$

Suppose, then, (67) yields a value of frequency ω for the trial function where

$$\omega=\omega_0+\Delta. \quad (70)$$

Then, it can be shown that Δ is of the second order in ϵ_i. Thus (67) is a variational form of the solution.

ACKNOWLEDGMENT

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Microphony in Waveguide*

I. GOLDSTEIN† AND S. SOORSOORIAN†

Summary—**This paper describes the mechanism of phase modulation by waveguide in the presence of a high intensity acoustic field. *X*-band rectangular was studied to determine the following:**

a) Resonant frequency in a transverse vibrational mode.
b) Means of minimizing phase modulation.

A CW radar can be represented as a microwave bridge in which the transmitted signal is compared in frequency with the received signal so that Doppler information may be obtained. Any disturbance of the bridge at the Doppler frequency will cause degradation in system sensitivity. It is our purpose to show that waveguide under a high acoustical field can definitely contribute to microphonics via the mechanism of phase modulation. This can be accomplished in many ways to a waveguide but we are primarily interested in transverse motion. The different transverse modes for the top and side waveguide walls are shown in Fig. 1.

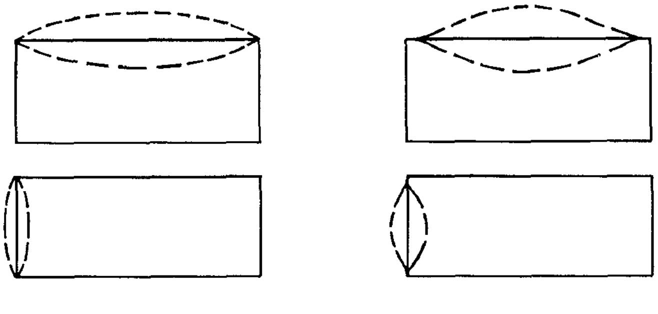

Fig. 1—First-mode vibration shapes.

Phase shift is accomplished by the motion of the side walls. The incremental phase shift is expressed as the following relationship for a rectangular waveguide operating in the TE_{10} mode:

$$d\phi=\frac{\pi l\lambda_g da}{2a^3},$$

$d\phi$ = incremental change of phase,
da = incremental change of side wall,
a = wide dimension of the waveguide,
l = length of the waveguide,
λ_g = guide wavelength.

Fig. 1 indicates an ideal situation of no coupling between waveguide walls. However, in actual practice there is coupling between the motion of the wide walls

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